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THE USE OF STRAIN GRADIENT THEORY FOR ANALYSIS OF RANDOM MEDIA

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Abstract—In this paper we consider the equations governing the response to a forcing field of an infinite statistically homogeneous medium with small fluctuations in the constants of elasticity. These equations were previously derived by Beran and McCoy [1]. The solution is obtained for the problem of a point force in an infinite medium and an analysis is presented to determine the ability of first strain gradient theory to approximate this solution. It is shown that a valid approximation can be obtained for the "slowly" varying (in space) portion of the solution only if the square of the lengths introduced in gradient theory for an isotropic centrosymmetric material are negative. This requirement violates the requirement that the strain energy density of the gradient theory be positive definite. Further consequences of choosing these lengths to be imaginary are considered. The value of going to higher order gradient theories is also discussed.

INTRODUCTION

IN A recent paper, Beran and McCoy [1], showed that the equations governing an ensemble of infinite random media are

$$\partial_i\{\tau_{ij}\}+F_j = 0,$$

$$\{\tau_{ij}(\mathbf{x})\} = \int \mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1)\{\varepsilon_{kl}(\mathbf{x}_1)\}\,\mathrm{d}\mathbf{x}_1, \tag{1}$$

$$\{\varepsilon_{kl}\} = \tfrac{1}{2}(\partial_k\{u_l\}+\partial_l\{u_k\})$$

$$\mathscr{C}_{ijkl}(\mathbf{x},\mathbf{x}_1) = [\{C_{ijkl}\}+D_{ijkl}(\mathbf{x}_1)]\delta(\mathbf{x}-\mathbf{x}_1)+E_{ijkl}(\mathbf{x},\mathbf{x}_1).$$

Here τ_{ij}, ε_{ij}, u_i are the stress, strain and displacement fields, respectively. F_j is a non-random forcing field. The braces indicate an ensemble average. C_{ijkl} is the elastic moduli tensor. This function is taken to be a random function of position. $D_{ijkl}(\mathbf{x}_1)$ and $E_{ijkl}(\mathbf{x},\mathbf{x}_1)$ are functions determined by the statistical properties of C_{ijkl} and the free space Green's function of the non statistical problem.

When the fluctuations in C_{ijkl} are small compared to the mean values $\{C_{ijkl}\}$, then

$D_{ijkl}(\mathbf{x})$ and $E_{ijkl}(\mathbf{x}, \mathbf{x}_1)$ may be evaluated explicitly. This has been done for the case of a locally isotropic media; i.e.

$$C_{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ij}\delta_{kl} + \mu(\mathbf{x})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}); \tag{2}$$

where $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are statistically homogeneous and isotropic random functions of position. The results, which are given by equations (30) and (31) in Ref. [1], are too lengthy to warrant reproducing them here.

Further in Ref. [1], it was argued that for those problems in which it is possible to identify two widely different length scales, it is permissible to invoke an ergodic hypothesis equating an ensemble average to a volume average. The two length scales are defined according to the following: On one scale it is possible to discern details of the variations of $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$. On this scale (the inner scale) the overall dimensions of the solid and any characteristic length associated with the forcing of the solid appear to be infinitely large. One the second scale (the outer scale) one can make measurements of the overall dimensions of the body and of characteristic lengths associated with the forcing function. On this scale the fluctuations in the material properties with position in space are too rapid to be discernable. Assuming that the two scales are present then an ergodic hypothesis allows one to equate the ensemble average to a volume average. The volume average is taken over a region that appears very small when viewed from the outer scale. Thus, one can infer from the statistical averages, information of the response of the individual solids that comprise the statistical sample. In the paper we consider only those problems in which the necessary conditions are present and interpret the braced quantities as denoting local volume averages.

The formulation given by equation (1) is not a unique one. Several investigators, including Kroner [2, 3], Krumhansl [4] and Kunin [5], interested in bridging the gap between the mass point approach to the mechanical behavior of solids and the continuum approach, have either begun with or arrived at the same formulation. The goal in these latter investigations was to develop a "non-local" elasticity theory; i.e. one which takes into account the finite range of the cohesive forces. The guiding light in these developments is crystal lattice theory. Thus, while the formulations are the same, the class of problems to which they are applicable are different.

The close relationship between the formulation expressed in equation (1) and the multipolar elasticity theory of the first kind proposed by Green and Rivlin [6] is readily apparent. The point of contact with the latter theory is achieved by expanding the strain tensor, $\{\varepsilon_{kl}(\mathbf{x}_1)\}$, as it appears in the constitutive equation of (1), as a power series about the point $\mathbf{x}$ and then carrying out the integration over $\mathbf{x}_1$ space. The result is a constitutive equation which relates the stress with the displacement gradients of all orders. Application of the Green and Rivlin theory requires our being able to truncate the series appearing in the constitutive equations after a relatively few terms. The result of such a set of truncations may be identified with a set of phenomenological theories, starting with the classical elasticity theory, then the first strain gradient theory of Toupin [7], the second strain gradient theory of Mindlin [8], etc. It is to be hoped that these theories of ever increasing complexity represent models of some problem with greater and greater fidelity. It is the purpose of the present paper to give some insight into this question for the class of problems which may be handled "exactly" by the formulation given by equations (1). This will be done by considering the response of an infinite solid to a concentrated force both within the framework of the integral formulation and within the framework of first strain gradient theory.

FORCED PROBLEM IN FOURIER SPACE

Substitution of the third of equations (1) into the second and the result into the first gives the equations governing the average displacement field. These equations of motion are seen to be three simultaneous Fredholm integro-differential equations of the second kind. For the case of homogeneous statistics, the integrals appearing in these equations are of the Faltung type suggesting the possible advantage of giving the equations a Fourier representation. The result is a set of three simultaneous algebraic equations on the transforms of the components of the average displacement field. This may be written

$$(\hat{\mathscr{C}}_{irsj}k_r k_s)\{\hat{u}_j\} = \hat{F}_i, \tag{3}$$

where k_i denotes the Fourier transform vector and $\hat{}$ denotes the transform of the indicated quantity.

For isotropic statistics, the coefficient matrix may be expressed by

$$(\hat{\mathscr{C}}_{irjs}k_r k_s) = \hat{f}_1 k_i k_j + \hat{f}_2 k^2 \delta_{ij}, \tag{4}$$

where $\hat{f}_1$ and $\hat{f}_2$ are functions of k alone.

It is a simple matter to invert equations (3), once equations (4) have been introduced. The result gives

$$\{\hat{u}_j\} = \left(\frac{\delta_{ij}}{k^2 \hat{f}_2} - \frac{\hat{f}_1}{\hat{f}_2(\hat{f}_1+\hat{f}_2)}\frac{k_i k_j}{k^4}\right)\hat{F}_i. \tag{5}$$

Equation (5) is essentially the desired solution for any forcing in transformed space.

In the limit of small fluctuations of $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ about their mean values it is possible to obtain explicit expressions for $\hat{f}_1(k)$ and $\hat{f}_2(k)$. These expressions, which are given by equations (50) and (51) of Ref. [1] are reproduced here for future reference.

$$\begin{aligned}
\hat{f}_1(k) &= \lambda^* + \mu^* + \frac{1}{\{\lambda+2\mu\}}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}\left[7M_3(kr) + \frac{\{\lambda+\mu\}}{\{\mu\}}M_7(kr)\right]\mathrm{d}r \\
&\quad + \frac{4}{\{\lambda+2\mu\}}\int_0^\infty \frac{C_{\lambda\mu}(r)+C_{\mu\lambda}(r)}{r}M_3(kr)\,\mathrm{d}r \\
\hat{f}_2(k) &= \mu^* + \frac{1}{\{\lambda+2\mu\}}\int_0^\infty \frac{C_{\mu\mu}(r)}{r}\left[M_3(kr) + \frac{3\{\lambda+\mu\}}{\{\mu\}}M_8(kr)\right]\mathrm{d}r,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
M_3(y) &= \left(\frac{1}{y} - \frac{3}{y^3}\right)\sin y + \frac{3}{y^2}\cos y, \\
M_7(y) &= \left(\frac{11}{y} - \frac{465}{y^3} + \frac{1080}{y^5}\right)\sin y + \left(\frac{105}{y^2} - \frac{1080}{y^4}\right)\cos y, \\
M_8(y) &= \left(-\frac{1}{y} + \frac{51}{y^3} - \frac{120}{y^5}\right)\sin y + \left(-\frac{11}{y^2} + \frac{120}{y^4}\right)\cos y,
\end{aligned} \tag{7}$$

and

$$\lambda^* = \{\lambda\} - \frac{\{\lambda'^2\} + \frac{4}{3}\{\lambda'\mu\} - \frac{4\{\lambda+\mu\}}{15\{\mu\}}\{\mu'^2\}}{\{\lambda+2\mu\}},$$

$$\mu^* = \{\mu\} - \frac{2\{3\lambda+8\mu\}}{\{\mu\}\{\lambda+2\mu\}}\{\mu'^2\}. \tag{8}$$

In the above, a prime denotes fluctuations of the indicated quantity about its mean value. The definition of $C_{\lambda\mu}(r)$ is

$$C_{\lambda\mu}(r) = \{\lambda'(0)\mu'(\mathbf{r})\}, \tag{9}$$

and similar expressions are given for $C_{\mu\lambda}$ and $C_{\mu\mu}$. The definitions of λ^* and μ^* are chosen so that in the limit of $k \to 0$,

$$C_{ijkl}(k) \to \lambda^*\delta_{ij}\delta_{kl} + \mu^*(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{10}$$

Since $k = 0$ corresponds to the case in which the average stress and strain fields are homogeneous, it is natural to denote this limit as the effective elastic moduli tensor. (For a detailed discussion of effective constants from a statistical point of view see Beran [9].)

Of particular interest in this investigation is the behavior of $\hat{f}_1$ and $\hat{f}_2$ in the vicinity of $k = 0$; i.e. $kl \ll 1$, where l is a characteristic length defined by variations in $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$. It is the contributions from this region of k space that one hopes to approximate by the various gradient theories. Expanding $\hat{f}_1(k)$ and $\hat{f}_2(k)$ about $k = 0$, and retaining two terms, gives

$$(\hat{f}_1 + \hat{f}_2) = (\lambda^* + 2\mu^*)(1 - k^2 l_1^2),$$

and

$$\hat{f}_2 = \mu^*(1 - k^2 l_2^2), \tag{11}$$

where

$$\{\lambda+2\mu\} l_1^2 = \frac{8}{15}\sigma_{\lambda\mu} + \frac{8\{\lambda+8\mu\}}{105\{\mu\}}\sigma_{\mu\mu},$$

and

$$\{\lambda+2\mu\} l_2^2 = \frac{\{3\lambda+10\mu\}}{105\{\mu\}}\sigma_{\mu\mu}. \tag{12}$$

In the above

$$\sigma_{\lambda\mu} = \int_0^\infty r C_{\lambda\mu}(r)\,\mathrm{d}r,$$

$$\sigma_{\mu\mu} = \int_0^\infty r C_{\mu\mu}(r)\,\mathrm{d}r$$

$\sigma_{\mu\mu}$ is non-negative. This is true since

$$\Phi_2(\mathbf{k}) = \int_0^\infty \int_0^{2\pi} C_{\mu\mu}(r)e^{i\mathbf{k}.\mathbf{r}} r \, dr \, d\phi$$

is non-negative for all values of **k** including $\mathbf{k} = 0$. $\sigma_{\lambda\mu}$ may be positive or negative.

l_2^2 is thus non-negative. l_1^2 may be positive or negative. If $\sigma_{\mu\mu} > |\sigma_{\lambda\mu}|$, however, l_1^2 is also always positive. For purposes of this paper we shall, for convenience, assume that this condition is met and that $l_1^2 > 0$. The fact that l_2^2 alone is positive allows us to draw all the conclusions we arrive at in this paper and choosing l_1^2 to be positive is thus not an essential restriction. We do note, however, that if $\sigma_{\mu\mu} = 0$ this is not an appropriate assumption. We exclude this special case from our considerations here and assume $\sigma_{\mu\mu} > 0, l_2^2 > 0$.

The linearized version of the equations of motion of Toupin's strain gradient theory are presented by Mindlin and Eshel [10] as

$$(\lambda+2\mu)(1-l_{1s}^2\nabla^2)\nabla\nabla \,.\, \mathbf{u}-(1-l_{2s}^2\nabla^2)\nabla\times\nabla\times\mathbf{u}+\mathbf{F} = 0. \tag{13}$$

In equation (13), λ and μ are the Lamé constants while l_{1s} and l_{2s} are the two additional constants required for an isotropic centrosymmetric solid. It is noted by Mindlin and Eshel that the requirement of positive definiteness of the strain energy density of the strain gradient theory requires that $l_{1s}^2 > 0$ and $l_{2s}^2 > 0$.

Casting equation (13) into Fourier space and solving the resulting algebraic equations for $\hat{\mathbf{u}}$ reproduces equation (5) with $\hat{f}_1$ and $\hat{f}_2$ replaced by $\hat{f}_{1s}$ and $\hat{f}_{2s}$, where

$$(\hat{f}_{1s}+\hat{f}_{2s}) = (\lambda+2\mu)(1+l_{1s}^2k^2),$$

and

$$\hat{f}_{2s} = \mu(1+l_{2s}^2k^2). \tag{14}$$

Thus, we see that the strain gradient solution in Fourier space agrees with the integral formulation solution in Fourier space in the vicinity of $k = 0$ provided one can identify $l_{1s}^2 = -l_1^2$ and $l_{2s}^2 = -l_2^2$. As we have already pointed out, however, l_1^2 and l_2^2 are positive numbers. Hence if strain gradient theory is to represent a valid approximation in the vicinity of k small, then it is necessary to take l_{1s}^2 and l_{2s}^2 to both be less than zero. This, of course, violates the requirement of positive definiteness of the strain energy density and is somewhat unsatisfactory.

Finally, it might be noted that the difficulty of violating the positive definiteness of the strain energy density is not unique to this problem. Mindlin [11] encountered the same difficulty in forcing the dispersion relations for waves predicted by the strain gradient theory to agree in the long wave length limit with the exact dispersion relations for a simple cubic Bravais lattice. His comments regarding this difficulty appear to be in agreement with those made here.

SOLUTION IN PHYSICAL SPACE

In this section we should like to consider the inversion of equation (5) to physical space. In particular we shall be interested in the fundamental problem of a concentrated force. We note that for such a forcing field $\{u_j\}$ is a function of k alone and hence the integration over the angular coordinates of **k** may be readily carried out. To this end we note

$$\int_s e^{i\mathbf{k}.\mathbf{r}}\, d\mathbf{k}_s = 4\pi k \frac{\sin kr}{r}, \tag{15}$$

where the surface integral is carried over a sphere of radius k. The result of the integration over angular coordinates gives

$$2\pi^2 u_i(r) = F_i \int_0^\infty \frac{1}{\hat{f}_2} \frac{\sin kr}{kr} dk + F_j \int_0^\infty \frac{\hat{f}_1}{\hat{f}_2(\hat{f}_1+\hat{f}_2)k^2} \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{\sin kr}{kr}\right) dk. \tag{16}$$

The indicated integrals can be evaluated by the process of analytic continuation and contour integration. Without going into excessive details on these well known techniques we first note that since $\hat{f}_1$ and $\hat{f}_2$ are even functions of k we may write

$$2\pi^2 u_i(r) = F_i \frac{\text{Im}}{2} \int_C \frac{1}{\hat{f}_2} \frac{e^{ikr}}{kr} dk + {}_1F_j \frac{\text{Im}}{2} \int_C \frac{\hat{f}_1}{\hat{f}_2(\hat{f}_1+\hat{f}_2)k^2} \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{e^{ikr}}{kr}\right) dk, \tag{17}$$

where Im denotes the imaginary part of the indicated quantity and C denotes k ranging from $-\infty$ to ∞. It might be noted that the integrals appearing in (14) will exist only in a principal value sense. We next view k as a complex variable and extend the definitions of the functions of k for values of k not on the real axis by the process of analytic continuation. The integrals may now be evaluated by means of residue theory by completing the contour in the usual manner. Under a suitable assumption regarding the behavior of $\hat{f}_1$ and $\hat{f}_2$ as k approaches infinity one can conclude that the appropriate line integrals taken about a large circle in the upper half-place vanish as the radius of the circle increases without bound. Thus, such a contour, together with any detour needed to exclude singularities of the integrands, makes a suitable closing contour. The value of the integral over C is now determined by the integrals taken over the detours introduced by the singularities. The integrals taken over detours introduced by pole type singularities are readily obtained by residue theory. Turning to the specific integrals to be evaluated, both integrands have pole type singularities at $k = 0$ plus any singularities introduced by the definitions of $\hat{f}_1$ and $\hat{f}_2$.

Strain gradient theory—l^2 is positive

Classical elasticity theory gives $\hat{f}_1 = \lambda+\mu$ and $\hat{f}_2 = \mu$, i.e. two analytic functions of k. Thus, the only singularities appearing in the integrands of (14) are the pole type singularities at $k = 0$. Evaluating the residues at $k = 0$ for this choice of $\hat{f}_1$ and $\hat{f}_2$ allows us to construct the classical Kelvin solution. This solution contains terms which vary as r^{-1} and r^{-3} where r denotes the distance from the point of application of the force to the field point. Turning to the strain gradient approximation with l_{1s}^2 and l_{2s}^2 both positive, we can note two effects. The residue of the second integrand in equations (14) for the singularity at $k = 0$ differs from the residue calculated for the classical elasticity approximation. The second is the introduction of pole type singularities at $k = \pm i/l_{2s}$ for the first integrand and at $k = \pm i/l_{1s}$ for the second integrand. It is the poles in the upper half-plane which offer contributions and the most important characteristic of these contributions, from our point of view, is that they decay exponentially with distance from the point of application of the force. The rate of decay is determined by $1/l_{1s}$ and $1/l_{2s}$. The actual solution to the concentrated force problem within the framework of strain gradient theory has already been presented by Mindlin [12].

Strain gradient theory—l_{is}^2 negative

Changing the sign of l_{1s}^2 and l_{2s}^2, which is required if strain gradient theory is to represent an approximation for the random medium problem, has the effect of changing the value of the residue at $k = 0$ and of moving the additional poles from the imaginary k axis to the real k axis. This second change gives rise to a marked change in the nature of the contribution to the solution that is a result of the presence of these poles. Now the contribution varies sinusoidally with distance from the point of application of the force. The periods of spatial oscillations are determined by l_{1s} and l_{2s}. The next question to be answered pertains to the value of the solution predicted by strain gradient theory with negative l_{1s}^2 and l_{2s}^2. (As far as the random medium problem is concerned the solution for positive l_{1s}^2 and l_{2s}^2 can have no validity.) It might be expected to give a valid correction to the residue at the pole at $k = 0$. This is simply because $\hat{f}_1$ and $\hat{f}_2$, as predicted by strain gradient theory, is really a two term expansion of the correct $\hat{f}_1(k)$ and $\hat{f}_2(k)$ about $k = 0$ while classical elasticity theory gives only a single term expansion. On the other hand, the contributions from the singularities at $k = \pm{}^{i}/l_{1s}$ and $k = \pm{}^{i}/l_{2s}$ must surely be suspect. If l_{1s} and l_{2s} represent a measure of the spatial variations of $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ then the truncation of the expansion of $\hat{f}_1(k)$ and $\hat{f}_2(k)$ after only two terms is valid only for those k which satisfy the inequality $kl_{1s} \ll /$ and $kl_{2s} \ll /$. The k's which locate these new poles obviously do not satisfy this inequality.

Second gradient theory

We next consider the value of going to a second gradient theory, the second gradient theory reproduces equations (5) but now $\hat{f}_1$ and $\hat{f}_2$ contain k^4 terms. Viewed as a small k expansion of the exact integral theory, the coefficients of the k^4 terms can be explicitly calculated for the medium in which the magnitudes of the fluctuations in $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are small relative to their mean values. If we consider now the effects of keeping k^4 terms we come to the following conclusions. First of all, no new modifications are introduced to the residues for the poles at $k = 0$. Only the values of their second derivatives with respect to k at $k = 0$ affect these residues. Secondly, the positions of the added poles of first strain gradient theory are shifted. Finally, additional poles will be introduced. Presumably, the contributions from these last poles will decay exponentially with distance, r, at a faster rate than will the exponentially decaying terms of first strain gradient theory. The value of any of the changes introduced by the second gradient theory is, therefore, questionable since the shift in position of poles that it predicts is in a region of k space in which truncating after k^4 terms is not valid.

Exact solution

Finally, we should like to turn to the exact expressions for $\hat{f}_1(k)$ and $\hat{f}_2(k)$ as given by equations (6)–(9). These expressions are derived for the limit of small perturbations in λ' and μ'. We should like to indicate that the poles defined by the zeroes of $\hat{f}_2$ and $\hat{f}_1 + \hat{f}_2$ cannot be on the real axis. Thus, the contributions to the solution that results from these poles must decay exponentially with r. The purely oscillatory terms predicted by first strain gradient theory with negative l_{1s}^2 and l_{2s}^2 are, therefore, spurious.

The form of $\hat{f}_1$ and $\hat{f}_2$ are such that

$$\hat{f}_1+\hat{f}_2 = \{\lambda+2\mu\}+g(k),$$
$$\hat{f}_2 = \{\mu\}+h(k), \tag{18}$$

where it is expected that

$$|g(k)| \ll \{\lambda+2\mu\}$$
$$|h(k)| \ll \{\mu\} \tag{19}$$

for k real. The reasoning behind this statement is the multiplicative factors of order $\{\lambda'^2\}^{\frac{1}{2}}$ and $\{\mu'^2\}^{\frac{1}{2}}$ for $g(k)$ and $h(k)$. By direct calculation we can show the inequalities are valid for $k \to 0$ and $k \to \infty$. We assume that $C_{\lambda\mu}$, $C_{\mu\lambda}$ and $C_{\mu\mu}$ are sufficiently well behaved to insure validity of the inequality for intermediate value of k on the real axis. If the inequalities given in equation (19) are satisfied $\hat{f}_1+\hat{f}_2$ and $\hat{f}_2$ cannot be zero for real values of k and therefore there are no oscillatory solutions.

CONCLUSION

In summary, this paper presents an analysis of the ability of the strain gradient theories to approximate the solution of the problem of a point force in an infinite random medium. The exact solution was determined from the integral formulation given by equation (1). It was seen that the exact solution can be subdivided into two parts. The first part has terms which vary spatially as r^{-n}, $n \geq 1$, might be termed the modified Kelvin solution. The second part has terms which decay exponentially with distance r. This part might be termed the force layer solution. With this subdivision, it was possible to conclude that the modification to the Kelvin solution introduced by strain gradient theory with a correct choice of l_{1s}^2 and l_{2s}^2 brings it into agreement with the modification that would be introduced by the full integral theory. On the other hand, the prediction of strain gradient within the force layer is completely spurious. Indeed if l_{1s}^2 and l_{2s}^2 are chosen so as to bring about the correct modification of the Kelvin solution, then the attendant force layer solution varies sinusoidally rather than exponentially. In order to obtain the correct solution using strain gradient theory it was necessary to give up the positive definiteness of the strain energy density.

Further, it was shown that going to a higher order gradient theory for which the additional material parameters are determined by expanding the integral function of the constitutive equation into an infinite sum of differential functions and then truncating, offers no improvement to the force layer solutions. An improvement could be achieved in, say, going to the second gradient theory if the additional constants of second gradient theory; i.e. additional to those of the first gradient theory; were chosen so as to bring the locations of those zeroes of $\hat{f}_2$ and $\hat{f}_1+\hat{f}_2$, as predicted by second gradient theory, into their best position. The best position is defined as that position which is closest to the position of these same zeroes as predicted by the exact $\hat{f}_2$ and $\hat{f}_1+\hat{f}_2$. Since it is the location of these poles which dominate the behavior in the "thickest" force layer, such a procedure might allow the region of validity of the second gradient theory to penetrate this layer.

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Абстракт—В работе исследуется уравнения, определяющие реакцию на вынуждающее поле бесконечной статистически однородной среды с малыми флуктуациями постоянных упругости. Зти уравнения выведены заранеее Бераном и Мак Койем. Приводится решение задачи для сосредоточенной силы в бесконечной среде. Дается адализ для определения способности теории градиента деформации первого пирчуки для аппроксимации зтого решения. Указывеатся, что применимое к данному случаю приближение чечко получать для "медленного" изменения /в пространстве/ части решения только тогда, когда квадраты длин введенны в теорию градиента являются отрицатедрными для осесимметрических материалов. Это требование нарушает требование, что пдотность знергии деформации теории градиента определена положительно. Исследуются дальшие последствия выбора зтих длин, которые оказываются мнимыми. Обсуждается также величина перехода к теориям градиента высших ицряукцг.